MATH 2850: UNIT STEP FUNCTIONS

DEFINITION: The **unit step** function $\mathcal{U}(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a \end{cases}$.

EXAMPLE: Graph the following:

- $f(t) = \mathcal{U}(t-1)$

- $f(t) = t^2\mathcal{U}(t-1)$

- $f(t) = (t-1)^2\mathcal{U}(t-1)$

- $f(t) = t^2\mathcal{U}(t) - t^2\mathcal{U}(t-2)$

- $f(t) = t^2\mathcal{U}(t) - t^2\mathcal{U}(t-2) + 4\mathcal{U}(t-2)$

EXAMPLE: Write in terms of Unit Step Functions:

- A **square wave**: for $A > 0$, $f(t) = \dfrac{1}{A}$ for $0 \leq t < A$

Ans: $f(t) = \dfrac{1}{A}\left(\mathcal{U}(t) - \mathcal{U}(t-A)\right)$

- $f(t) = \begin{cases} 3 & \text{if } 0 \leq t < 1 \\ 2t & \text{if } 1 \leq t < 3 \\ 9 - t^2 & \text{if } 3 \leq t < 5 \end{cases}$

Ans: $f(t) = 3\left(\mathcal{U}(t) - \mathcal{U}(t-1)\right) + 2t\left(\mathcal{U}(t-1) - \mathcal{U}(t-3)\right) + (9 - t^2)\left(\mathcal{U}(t-3) - \mathcal{U}(t-5)\right)$

THEOREM: Laplace Transforms and Unit Step Functions:

- $\mathcal{L}\left\{\mathcal{U}(t-a)\right\} = \dfrac{e^{-as}}{s}$; in particular, $\mathcal{L}\left\{\mathcal{U}(t)\right\} = \dfrac{1}{s}$
- $\mathcal{L}\left\{\mathcal{U}(t-a)f(t-a)\right\} = e^{-as}F(s)$
- $\mathcal{L}\left\{\mathcal{U}(t-a)\, f(t)\right\} = e^{-as}\mathcal{L}\{f(t+a)\}$

PROOF:

EXAMPLE: Find the following Laplace Transforms:

- $\mathcal{L}\left\{(t-2)^2\mathcal{U}(t-2)\right\}$

Ans: $\mathcal{L}\left\{(t-2)^2\mathcal{U}(t-2)\right\} = \dfrac{2e^{-2s}}{s^3}$

- $\mathcal{L}\left\{t^2\mathcal{U}(t-2)\right\}$

Ans: $\mathcal{L}\left\{t^2\mathcal{U}(t-2)\right\} = e^{-2s}\left(\dfrac{2}{s^3}+\dfrac{4}{s^2}+\dfrac{4}{s}\right)$

- $\mathcal{L}\{f(t)\}$ where $f(t)$ is a square wave: for $A > 0$, $f(t) = \dfrac{1}{A}$ for $0 \le t < A$

Ans: $\mathcal{L}\{f(t)\} = \dfrac{1}{A}\left(\dfrac{1}{s} - \dfrac{e^{-As}}{s}\right)$

- $\mathcal{L}\{f(t)\}$ where $f(t) = \begin{cases} 3t+1 & \text{if } 0 \le t < 1 \\ \sin(\pi t) & \text{if } 1 \le t < 2 \end{cases}$

Ans: $\mathcal{L}\{f(t)\} = \dfrac{3}{s^2} + \dfrac{1}{s} - \dfrac{3e^{-s}}{s^2} - \dfrac{4e^{-s}}{s} - \dfrac{e^{-s}\pi}{s^2+\pi^2} - \dfrac{e^{-2s}\pi}{s^2+\pi^2}$

THEOREM: Backward Exponential Shift: $\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\} = \mathcal{U}(t-a)\mathcal{L}^{-1}\left\{F(s)\right\}_{t\to(t-a)} = \mathcal{U}(t-a)f(t-a)$

EXAMPLE: Find the following Inverse Laplace Transforms:

- $\mathcal{L}^{-1}\left\{\dfrac{1}{s^5e^{3s}}\right\}$

Ans: $\mathcal{L}^{-1}\left\{\dfrac{1}{s^5e^{3s}}\right\} = \dfrac{(t-3)^4\,\mathcal{U}(t-3)}{24}$

- $\mathcal{L}^{-1}\left\{\dfrac{e^{-2s}}{s(s+2)}\right\}$

Ans: $\mathcal{L}^{-1}\left\{\dfrac{e^{-2s}}{s(s+2)}\right\} = \dfrac{1}{2}\mathcal{U}(t-2)\,\left(1-e^{-2t+4}\right)$

- $\mathcal{L}^{-1}\left\{\dfrac{e^{-\pi s}}{s^2+1}\right\}$

Ans: $\mathcal{L}^{-1}\left\{\dfrac{e^{-\pi s}}{s^2+1}\right\} = -\mathcal{U}(t-\pi)\,\sin(t)$

- $\mathcal{L}^{-1}\left\{\dfrac{2se^{-\pi s}}{s^2+2s+10}\right\}$

Ans: $\mathcal{L}^{-1}\left\{\dfrac{2se^{-\pi s}}{s^2+2s+10}\right\} = \dfrac{2}{3}\,e^{-(t-\pi)}\,\mathcal{U}(t-\pi)\,(\sin(3t)-3\cos(3t))$

HOMEWORK: Section 8.4: Pg. 428: 1 - 27 odd.